

Reconstruction of Bandlimited Functions from Unsigned Samples

Gaurav Thakur*

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Abstract

We consider the recovery of real-valued bandlimited functions from the absolute values of their samples, possibly spaced nonuniformly. We show that such a reconstruction is always possible if the function is sampled at more than twice its Nyquist rate, and may not necessarily be possible if the samples are taken at less than twice the Nyquist rate. In the case of uniform samples, we also describe an FFT-based algorithm to perform the reconstruction. We prove that it converges exponentially rapidly in the number of samples used and examine its numerical behavior on some test cases.

Keywords: Sampling theorem, Nonuniform sampling, Entire functions of exponential type, Canonical products, Fast Fourier Transform

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1 Introduction

In a series of recent papers [1, 2, 3], Balan, Casazza and Edidin have investigated the possibility of reconstructing finite-dimensional signals using measurements that do not contain sign or phase information. Motivated by an application in the denoising of speech signals, they studied *M-element frames* for \mathbb{R}^n , i.e. collections of M vectors that span \mathbb{R}^n . They considered *M-element frames*, $\{f_k\}_{1 \leq k \leq M}$, $f_k \in \mathbb{R}^n$, such that any vector $x \in \mathbb{R}^n$ can be uniquely determined from the inner products $\{|\langle f_k, x \rangle|\}_{1 \leq k \leq M}$ up to an ambiguity of a sign factor. Using frame theory and combinatorial methods, they showed that such frames exist if and only if $M \geq 2n + 1$. In [1], a computational method to carry out this reconstruction was described in the case where $M \geq \frac{n(n+1)}{2}$, using a special class of frames.

It is natural to ask if there are analogous results for continuous-domain signals, namely in the context of samples of bandlimited functions. The well-known Whittaker-Shannon-Kotelnikov (WSK) sampling theorem [4] shows that if a bandlimited function f is sampled at a rate greater

*Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544, USA, email: gthakur@princeton.edu

than its Nyquist frequency, then it can be uniquely reconstructed from the samples. However, if the signs of the samples are unknown, this condition may no longer be sufficient. An example illustrating this is given by the functions $\sin(\pi(z + \frac{1}{4}))$ and $\cos(\pi(z + \frac{1}{4}))$, which agree in absolute value at $z = \frac{k}{2}$, $k \in \mathbb{Z}$. On the other hand, it might be expected that if we oversample f at a sufficiently high rate, the samples may contain enough redundancy that we can afford to lose their signs and still recover f up to a global sign factor. It turns out that this is indeed the case.

In this paper, we use a complex variable approach to show that if a real-valued bandlimited function f is sampled at more than twice its Nyquist rate, then f can be uniquely determined from the absolute values of its samples up to a sign factor. Conversely, we find that if f is sampled at less than twice its Nyquist rate, then it is not always possible to uniquely determine it in this way. We present an algorithm to perform this reconstruction, and show that it converges exponentially rapidly in the number of samples used. We consider a fairly general class of nonuniformly spaced samples in this paper, although our numerical approach is developed with uniformly spaced samples in mind for reasons of computational efficiency.

We review some existing theory on nonuniform sampling and bandlimited functions in Section 2, and then state and prove our main theoretical results in Section 3. We describe our algorithm and study its convergence properties in Section 4, and apply it to two test cases in Section 5.

2 Background Material

We normalize the Fourier transform as $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t} dt$ for Schwartz functions f and extend it to tempered distributions in the usual way. For $0 < p \leq \infty$, we define the *Paley-Wiener spaces* of bandlimited functions [14] by

$$PW_b^p = \{f \in L^p : \text{supp}(\hat{f}) \subset [-\frac{b}{2}, \frac{b}{2}]\}.$$

An entire function g is said to be of *exponential type b* if

$$b = \inf \left(\beta : |g(z)| \leq e^{\beta|z|}, z \in \mathbb{C} \right),$$

and we denote this by writing $\text{type}(g) = b$. By the Paley-Wiener-Schwartz theorem [8], PW_b^p can be equivalently described as the space of all entire functions f with $\text{type}(f) \leq \pi b$ whose restrictions to \mathbb{R} are in L^p . It also follows that $PW_b^p \subset PW_b^q$ for $p < q$. Functions $f \in PW_b^p$ satisfy the classical estimates $\|f'\|_{L^p} \leq \pi b \|f\|_{L^p}$ and $\|f(\cdot + ic)\|_{L^p} \leq e^{\pi b|c|} \|f\|_{L^p}$, respectively known as the *Bernstein* and *Plancherel-Polya inequalities* [11, 14].

We now consider a sequence of points $X = \{x_k\} \subset \mathbb{R}$, indexed so that $x_k < x_{k+1}$. For any set B , we denote the number of x_k in B by $N(X, B)$. We say that X is *separated* if $\inf_k |x_{k+1} - x_k| > 0$. Following [5], X is also said to be *uniformly dense* if it satisfies

$$\sup_k \left| x_k - \frac{k}{d} \right| < \infty \tag{1}$$

for some finite $d > 0$. We denote this by writing $D(X) = d$, and by $D(X) = \infty$ if X is not uniformly dense. We will mainly deal with separated, uniformly dense sequences in this paper. It is worth

mentioning that $D(X)$ is not directly related to Beurling's upper and lower densities (see [14]), and there are sequences X with finite Beurling densities but for which $D(X) = \infty$.

The *generating function* of a sequence $X \subset \mathbb{R}$ is given by

$$S(z) = z^{\delta_X} \lim_{r \rightarrow \infty} \prod_{0 < |x_k| < r} \left(1 - \frac{z}{x_k}\right), \quad (2)$$

where $\delta_X = 1$ if $0 \in X$ and $\delta_X = 0$ otherwise. For a uniform sequence $x_k = \frac{k}{b}$, $S(z) = \frac{\sin(\pi bz)}{\pi b}$. If X is separated and $D(X) = b < \infty$, then the limit in (2) is finite and the function S lies in the *Cartwright class* CW_b [9], the set of all entire functions f with $\text{type}(f) \leq \pi b$ that satisfy the growth condition

$$\int_{-\infty}^{\infty} \frac{\max(\log |f(t)|, 0)}{t^2 + 1} dt < \infty.$$

In fact, functions in CW_b satisfy the apparently stronger condition [10]

$$\int_{-\infty}^{\infty} \frac{|\log |f(t)||}{t^2 + 1} dt < \infty. \quad (3)$$

The following result shows that an arbitrary function in CW_b can be expanded in the form (2) and gives a useful geometric description of its zeros. [11]

Theorem. (*Cartwright-Levinson*) Let $W^+(\theta, r)$, $W^-(\theta, r)$ and $W'(\theta, r)$ respectively be the wedges $\{z : |z| < r, |\arg z| \leq \theta\}$, $\{z : |z| < r, |\pi - \arg z| \leq \theta\}$ and $\{z : |z| < r, |\arg z| > \theta, |\pi - \arg z| > \theta\}$. Suppose $f \in CW_b$, $f \not\equiv 0$, and let $U = \{u_k\}$ be the set of its zeros. Define $b' = \frac{1}{\pi} \inf(\text{type}(e^{i\omega z} f(z)), \omega \in \mathbb{R})$.

1: For any $\theta \in (0, \frac{\pi}{2})$, as $r \rightarrow \infty$,

$$\frac{N(U, W^+(\theta, r))}{r} \rightarrow \frac{b'}{2}, \quad \frac{N(U, W^-(\theta, r))}{r} \rightarrow \frac{b'}{2}, \quad \text{and} \quad \frac{N(U, W'(\theta, r))}{r} \rightarrow 0.$$

2: U satisfies $\sum_{|u_k| > 0} |\text{Im}(\frac{1}{u_k})| < \infty$.

3:

$$f(z) = f(0) e^{iqz} z^{\delta_U} \lim_{r \rightarrow \infty} \prod_{0 < |u_k| < r} \left(1 - \frac{z}{u_k}\right), \quad (4)$$

for some constant q .

If f is real-valued on \mathbb{R} , then $b' = b$ and $q = 0$, so in particular, the sequence of real zeros $V \subset U$ of f satisfies $\lim_{r \rightarrow \infty} \frac{N(V, [-r, r])}{r} \leq b$. The expression on the right side of (4) is called a *canonical product*.

We will also need a second, deeper theorem on CW_b .

Theorem. (Beurling-Malliavin) Let $f \in CW_b$. Then for any $\varepsilon > 0$, there exists $h \in CW_\varepsilon$, $h \neq 0$, such that $fh \in L^\infty$.

A discussion of this result and its significance can be found in [7]. We can actually choose the above h so that $fh \in L^p$ for any $p > 0$, by replacing h with $h\phi$ where ϕ is a function for which $\hat{\phi}$ is smooth and has sufficiently small support. By considering the function $h(z)\overline{h(\bar{z})}$, h can also be taken to be real and nonnegative.

For $1 \leq p \leq \infty$, X is called a *sampling sequence* for PW_b^p if there are constants $\alpha_{p,b}$ and $\beta_{p,b}$ such that for all $f \in PW_b^p$, $\alpha_{p,b}\|f(X)\|_{l^p} \leq \|f\|_{L^p} \leq \beta_{p,b}\|f(X)\|_{l^p}$. In particular, this condition implies that any $f \in PW_b^p$ is uniquely determined by its samples at X . Precise geometric characterizations of sampling sequences can be very complicated (see [4, 13, 14]), but for real, separated and uniformly dense sequences X , it is necessary that $D(X) \geq b$ and sufficient that $D(X) > b$ for X to be a sampling sequence. If $D(X) > b$ and S is the generating function of X , a consequence of the Beurling-Malliavin theorem is that there is an $h \in CW_\varepsilon$ with zero set U , $\varepsilon = D(X) - b$, such that any $f \in PW_b^\infty$ can be expressed in terms of its samples,

$$f(z) = \sum_{k=-\infty}^{\infty} f(x_k) \frac{S(z)h(x_k)}{h(z)S'(x_k)(z - x_k)}, \quad (5)$$

with uniform convergence on compact subsets of $\mathbb{C} \setminus U$ [9, 13]. $D(X)$ can be thought of as a generalization of the “sampling rate” to uniformly dense sequences X , and when $X = \{\frac{k}{s}\}$ is a uniform sequence, the condition $b < D(X) = s$ simply says that f is being oversampled beyond its Nyquist rate. h can be taken as constant for uniform sequences, and the expansion (5) reduces to the classical WSK sampling theorem.

3 Main Results

We fix $p \in (0, \infty]$ for the rest of this section. We first show that if a bandlimited function is sampled at more than twice its Nyquist rate, then we can reconstruct it up to a sign factor from the absolute values of the samples.

Theorem 1. Let $f \in PW_b^p$ be real-valued on \mathbb{R} , and let $X \subset \mathbb{R}$ be a separated, uniformly dense sequence with $D(X) > 2b$. Then f can be uniquely determined from $a_k = |f(x_k)|$, up to a sign factor.

Proof. We normalize $b = 1$ without loss of generality. Let S be the generating function of X and suppose h is given as in (5). The zeros of f and h are countable, so we can choose $c > 0$ so that f and h have no zeros on the line $L = \{z : \text{Im}(z) = c\}$. We can write $g = f(\cdot + ic)^2 \in PW_2^{p/2}$ as

$$g(z) = \sum_{k=-\infty}^{\infty} a_k^2 \frac{S(z+ic)h(x_k)}{h(z+ic)S'(x_k)(z+ic-x_k)}. \quad (6)$$

From Bernstein’s inequality, $g' \in PW_2^{p/2}$ and differentiating (6) gives a similar expansion for $g'(z)$. Let $a_k^* = f(x_k + ic)$ and choose a point $x_l \in X$. Since f has no zeros on L , there is a branch of $\arg f$, which we denote by $\arg_0 f$, that is continuous on L and satisfies $\arg_0 f(x_l + ic) \in (-\pi, \pi]$. We define $\arg_0 g$ in the same way, and we then have

$$a_k^* = \exp\left(\frac{1}{2}\log|g(x_k)| + i\arg_0 f(x_k + ic)\right) \quad (7)$$

$$= \eta |g(x_k)|^{1/2} \exp\left(\frac{i}{2}\left(\int_{x_l}^{x_k} \operatorname{Im}\left(\frac{g'(t)}{g(t)}\right) dt + \arg_0 g(x_l)\right)\right) \quad (8)$$

where $\eta = \exp\left(i\left(\frac{1}{2}\arg_0 g(x_l) - \arg_0 f(x_l + ic)\right)\right) = \pm 1$. We have now determined samples of f , which we can use to express f as

$$f(z) = \eta \sum_{k=-\infty}^{\infty} a_k^* \frac{S(z-ic)h(x_k)}{h(z-ic)S'(x_k)(z-ic-x_k)}. \quad (9)$$

□

Remark. The proof of Theorem 1 suggests a three-step procedure to recover f from $\{a_k\}$. We can determine f^2 from $\{a_k\}$ and take its square root by unwrapping its phase. Since f will typically have zeros on the real axis, we first move up in the complex plane using (6), unwrap the phase there with (8) and then move back to the real axis with (9). We will use this approach in Section 4.

The next result shows that Theorem 1 is in a sense sharp. If we sample a function at less than twice its Nyquist rate, it may or may not be uniquely determined by the absolute values of the samples, essentially depending on “how many” of the samples are zero.

Theorem 2. *Let $X \subset \mathbb{R}$ be a separated sequence with $D(X) < 2b$. Then there is a real-valued function $f \in PW_b^p$ that cannot be uniquely determined from $|f(x_k)|$ up to a sign factor. If in addition $D(X) > \frac{4}{3}b$, then there is another real-valued function $\tilde{f} \in PW_b^p$ that can be uniquely determined from $|\tilde{f}(x_k)|$ up to a sign factor, but for which there is no subsequence $Y \subset X$ with $\tilde{f}(y_k) = 0$ and $\lim_{r \rightarrow \infty} \frac{N(Y, [-r, r])}{r} \geq b$.*

Proof. As before, we normalize $b = 1$. Suppose $D(X) = 2 - \varepsilon$ for some $\varepsilon > 0$, and let $X_1 = \{x_{2k}\}$ and $X_2 = \{x_{2k+1}\}$. It follows from the definition (1) that $D(X_1) = D(X_2) = 1 - \frac{\varepsilon}{2}$. Now let S_1 and S_2 be the generating functions of X_1 and X_2 . By the Beurling-Malliavin theorem, we can find $h_1, h_2 \in CW_{\varepsilon/2}$ such that $S_1 h_1$ and $S_2 h_2$ are in PW_1^p . Then the functions $f_1 = S_1 h_1 + S_2 h_2$ and $f_2 = S_1 h_1 - S_2 h_2$ are also in PW_1^p and satisfy $|f_1(X)| = |f_2(X)|$.

For the other part of Theorem 2, suppose ε above satisfies $\varepsilon < \frac{2}{3}$. By countability, we can choose c so that $h_3(z) = h_1(z+c)$ has no zeros on X . Let $\tilde{f} = S_1 h_3$, so that among the samples of \tilde{f} at X , only the ones at X_1 are zero. If there is any real-valued $g \in PW_1^p$ with $|\tilde{f}(X)| = |g(X)|$, then $\tilde{f}^2 - g^2$ is in $PW_2^{p/2}$ and has zeros at X , so by considering canonical product expansions, $\tilde{f}^2 - g^2 = S_1 S_2 h_4$ for some $h_4 \in CW_\varepsilon$. Since $g^2 = S_1(S_1 h_3^2 - S_2 h_4)$ has an analytic square root, it can only have double zeros, so in particular, $S_1 h_3^2 - S_2 h_4$ has to be zero on X_1 . This implies that h_4 must have zeros on X_1 , which contradicts the Cartwright-Levinson theorem because $D(X_1) = 1 - \frac{\varepsilon}{2} > \varepsilon$. So $h_4 \equiv 0$ and $\tilde{f} = \pm g$. □

Remark. The nonexistence of Y in Theorem 2 is what makes the result interesting, as it means that \tilde{f} cannot be determined from Y by just using a canonical product. In other words, the nonzero samples at X_2 play a role in the uniqueness of \tilde{f} . However, there appears to be no simple characterization of all such functions \tilde{f} or a numerically useful method of computing \tilde{f} from $|\tilde{f}(X)|$.

Remark. We have not considered the border case of $D(X) = 2b$ in the above results, in which case the conditions required on the sequence X would become more subtle and depend on the value of p . However, in the elementary case where $x_k = \frac{k}{2b}$ is a uniform sequence and $p = 2$, the conclusion of Theorem 1 still holds by just using the WSK sampling theorem in place of (6) and (9).

There are no simple analogs of these results if we allow $f \in PW_b^p$ to be complex-valued. In Theorem 1, we used the fact that when f is real-valued, f^2 has the same samples as $|f|^2$, but this is no longer the case for complex-valued f . In general, such an f will have complex zeros u_k and complex-valued functions of the form Bf , where B is a Blaschke product formed from any subset of $\{\overline{u_k}\}$, will be in PW_b^p and satisfy $|Bf| = |f|$ identically on \mathbb{R} . If we require all the zeros of f to be real, then since f can be written as a canonical product over them, it is simply a modulation of a real-valued function g , i.e. it has the form $f(z) = e^{i(cz+d)}g(z)$ for $c, d \in \mathbb{R}$. This situation is in contrast to the findings in [2], where the types of frames the authors studied had results for complex vectors comparable to those outlined in Section 1 for real vectors.

4 A Reconstruction Algorithm

We now describe how to computationally implement the technique in the proof of Theorem 1. We restrict our attention to uniform sampling sequences here, as they lead to convolution-type sampling series that can be calculated efficiently by Fast Fourier Transform (FFT) methods, but the same ideas can be adapted to the nonuniform case. We first define the following functions:

$$\begin{aligned} G(z, M) &= \frac{\sin(\pi z)}{\pi z} e^{-\frac{\pi}{2M} z^2} \\ G'(z, M) &= \left(\frac{\cos(\pi z)}{\pi z} - \frac{\sin(\pi z)}{M} - \frac{\sin(\pi z)}{\pi z} \right) e^{-\frac{\pi}{2M} z^2} \\ G^*(z, M) &= \frac{1}{M} \int_{z-M}^z G(t, M) dt \end{aligned}$$

We also denote the strip $\{z : |\operatorname{Im}(z)| < \delta\}$ by T_δ . Then we have the following results from [12], reproduced here in slightly different forms for our purposes.

Theorem 3. (Schmeisser-Stenger, 2002) *Let $f \in PW_b^\infty$, $s > b$ and $d < 1$. Then for $\operatorname{Re}(z) \in [-dM, dM]$, $f(\frac{z}{s}) = \sum_{k=-M}^M f(\frac{k}{s}) G(z-k, M) + E(z)$, where $|E(z)| \leq C_1 M^{-1/2} e^{-\frac{\pi(1-d)}{2} (1-\frac{b}{s})M + 2\pi|\operatorname{Im}(z)|}$. $\|f\|_{L^\infty}$ for some constant $C_1 = C_1(\frac{b}{s})$.*

Theorem 4. (Schmeisser-Stenger, 2002) *Suppose f is analytic in T_δ and $|f(z)| \leq K|z|$ for $z \in T_\delta$ and some constant K . Then for $z \in \mathbb{R}$, $-\frac{d}{2} \leq z \leq \frac{d}{2}$, $f(z) = \sum_{k=-dM}^{dM} f(\frac{k}{M}) G(Mz-k, M) + E(z)$, where $|E(z)| \leq C_2 K (M/\delta)^{1/2} e^{-\frac{\pi\delta M}{4}} \|f\|_{L^\infty}$ for some constant C_2 .*

Theorem 3 is a form of (5) with a non-bandlimited h . It converges very rapidly in practice for about $s \geq 1.3b$. Theorem 4 is a version of Theorem 3 as $s \rightarrow \infty$.

These results lead to the following numerical approach. We impose a mild restriction to rule out some pathological functions, and assume for notational convenience that an odd number of samples are used.

Algorithm 5. Suppose $f \in PW_b^\infty$ is nonzero in some strip $\{z : |\operatorname{Im}(z) - c| \leq \delta\}$, $\delta > 0$. Let $s > 2b$ and $a_k = |f(\frac{k}{s})|$ for $-M \leq k \leq M$.

- 1: Compute $g_M(\frac{z}{s}) = \sum_{k=-M}^M a_k^2 G(z - k + ic, M)$ at $z = \frac{n}{M}$, $-M^2 \leq n \leq M^2$.
- 2: Compute $g'_M(\frac{z}{s}) = \sum_{k=-M}^M a_k^2 G'(z - k + ic, M)$ for z as above.
- 3: Compute $Q_M(n) = \frac{1}{s} \sum_{k=(n-2)M}^{(n+1)M} \operatorname{Im} \left(\frac{g'_M(\frac{k}{Ms})}{g_M(\frac{k}{Ms})} \right) G^*(Mn - k, M)$ for $-(M-2) \leq n \leq M-1$.
- 4: Compute $R_M(n)$ given by $R_M(0) = 0$, $R_M(n) = R_M(n-1) + Q_M(n-1)$ for $-(M-1) \leq n \leq M-1$.
- 5: Compute $f_M(\frac{z}{s}) = \sum_{k=-(M-1)}^{M-1} \sqrt{|g_M(\frac{k}{s})|} e^{\frac{i}{2}(R_M(k) + \arg g_M(0))} G(z - k - ic, M)$.

The convolutions in steps 1 and 2 can be performed by $2M$ FFTs of size $2M+1$ each. Steps 3 and 4 are most efficiently done by direct computations that respectively involve $O(M^2)$ and $O(M)$ operations. Step 5 involves $2N$ FFTs of size $2M+1$ for some integer N , depending on how finely we want to compute f . This gives an overall complexity of $O(M^2 \log M)$.

Theorem 4 is used above to calculate the integral in (8). The advantage of this approach is that the error bound in Theorem 4 does not depend on the derivatives of f . If we instead used conventional methods of integration such as Gauss quadrature, we would need additional restrictions on δ to ensure that the algorithm actually converges. There is no simple closed-form expression for G^* but its numerical calculation only depends on M , so we can tabulate its values by a standard quadrature method using $O(M^2)$ operations and reuse them for different f with a lookup table.

We can establish the following convergence result for Algorithm 5.

Theorem 6. Let f and f_M be defined as in Algorithm 5. As $M \rightarrow \infty$, $f_M \rightarrow \eta f$ uniformly on compact subsets of \mathbb{R} , where $\eta = \pm 1$ and $|f_M(z) - \eta f(z)| = O\left(e^{-\min(\frac{\pi}{16}(1-\frac{2b}{s}), \frac{\pi\delta}{8})M+4\pi c}\right)$.

This estimate is somewhat conservative and larger exponents of convergence are possible if we make more assumptions on f , but it shows how δ and c affect the rate of convergence. Many real-world bandlimited signals have most of their zeros on or near the real axis, so choosing c too small will result in a small δ while choosing c too large will sharply increase the constant in the error bound. Values of c between about 0.01 to 0.25 appear to work well in practice.

Before we prove Theorem 6, we will need an auxiliary lemma.

Lemma 7. Let $h \in PW_b^\infty$ be nonzero in T_δ . Then $\left| \frac{h'(z)}{h(z)} \right| \leq K(|z|^2 + 1)$ in $T_{\delta/2}$ for some constant K .

Proof. Let $U = \{u_k\}$ be the zeros of $h(z - i\delta)$ lying in the upper half plane $\mathbb{C}^+ = \{z : \text{Im}(z) > 0\}$, and suppose $\frac{\delta}{2} < \text{Im}(z) < \frac{3\delta}{2}$. The Plancherel-Polya inequality shows that $h(\cdot - i\delta) \in PW_b^\infty$ and that the function $e^{\pi ibz}h(z - i\delta)$ is bounded and analytic on \mathbb{C}^+ , so it has the inner-outer factorization [6]

$$\log h(z - i\delta) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1+zt}{(t-z)(t^2+1)} \log |h(t - i\delta)| dt + \log \left(\prod_k \frac{|u_k^2 + 1|}{u_k^2 + 1} \frac{z - u_k}{z - \bar{u}_k} \right) + Az + B$$

for some constants A and B . We differentiate this to find that

$$\begin{aligned} \left| \frac{h'(z - i\delta)}{h(z - i\delta)} \right| &= \left| \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\log |h(t - i\delta)|}{(t-z)^2} dt + \sum_k \frac{2i\text{Im}(u_k)}{(z - u_k)(z - \bar{u}_k)} + A \right| \\ &\leq (2/\delta)^2 (|\text{Re}(z)|^2 + 1) \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log |h(t - i\delta)|}{t^2 + 1} dt + \sum_k \frac{2\text{Im}(u_k)}{|u_k|^2 + 1} + |A| \right). \end{aligned}$$

Since $PW_b^\infty \subset CW_b$, condition (3) shows that the integral in the first term is finite. The Cartwright-Levinson theorem implies that the sum in the second term also converges, which finishes the proof. \square

It is possible to obtain sharper results than this, but this lemma is good enough for our purposes.

Proof of Theorem 6. We proceed by establishing several intermediate bounds and then combine them all at the end. To simplify the notation, we will always use $z \in \mathbb{R}$ and $k \in \mathbb{Z}$ to denote function arguments within norms, e.g. the norm of $F \in L^\infty(\mathbb{R})$ will be denoted by $\|F(z)\|$.

Let g and η be defined as in the proof of Theorem 1. Define $I_{r,M} = [-\lfloor r(M-1) \rfloor - 1, \lfloor r(M-1) \rfloor - 1]$, $i_{r,M} = \mathbb{Z} \cap I_{r,M}$ and $j_{r,M} = \mathbb{Z} \cap [-M \lfloor rM \rfloor, M \lfloor rM \rfloor]$. Theorem 3 shows that

$$E_1 := \left\| g_M\left(\frac{k}{Ms}\right) - g\left(\frac{k}{Ms}\right) \right\|_{L^\infty(j_{3/4,M})} \leq \frac{C_1}{\sqrt{M}} e^{-\frac{\pi}{8}(1-\frac{2b}{s})M+2\pi c} \|f\|_{L^\infty(\mathbb{R})}^2.$$

The error term E in Theorem 3 is an entire function and it satisfies the classical Cauchy estimate

$$\|E'(z + ic)\|_{L^\infty(I_{3/4,M+2})} \leq 2\pi \max \left(|E(z + ic)|, \left\{ z : |\text{Re}(z)| \leq \frac{3}{4}(M+1), |\text{Im}(z)| \leq \frac{1}{2\pi} \right\} \right).$$

This shows that

$$E_2 := \left\| g'_M\left(\frac{k}{Ms}\right) - g'\left(\frac{k}{Ms}\right) \right\|_{L^\infty(j_{3/4,M})} \leq \frac{2\pi C_1}{\sqrt{M}} e^{-\frac{\pi}{8}(1-\frac{2b}{s})(M-1)+2\pi c+1} \|f\|_{L^\infty(\mathbb{R})}^2.$$

A standard identity for Jacobi theta functions ([15], p. 475) gives the bound

$$E_3 := \left\| \sum_{l=-M}^M G(z - l - ic, M) \right\|_{L^\infty(I_{1/4,M})} \leq \left\| \sum_{l=-\infty}^{\infty} e^{\frac{\pi}{2M}(c^2 - (z-l)^2)} \right\|_{L^\infty(\mathbb{R})} \leq \frac{3}{2} e^{\frac{\pi c^2}{2M}} \sqrt{2M},$$

and similarly,

$$E_4 := \left\| \sum_{l=(k-2)M}^{(k+1)M} |G^*(Mk-l, M)| \right\|_{l^\infty(i_{3/4, M+2})} \leq \frac{3}{2} \sqrt{2M}.$$

Now by Lemma 7, $\left| \frac{g'(z)}{g(z)} \right| \leq K(|z|^2 + 1)$ in $T_{\delta/2}$ for a constant K . Since $\|g_M - g\|_{L^\infty(I_{3/4, M+2})} \rightarrow 0$ and g has no zeros on \mathbb{R} , we have $\|1/g_M\|_{L^\infty(I_{3/4, M+2})} < \infty$ for sufficiently large M . We also have the bound

$$\begin{aligned} \left\| \frac{g'_M(\frac{k}{Ms})}{g_M(\frac{k}{Ms})} - \frac{g'(\frac{k}{Ms})}{g(\frac{k}{Ms})} \right\|_{l^\infty(j_{3/4, M})} &\leq \left\| \frac{1}{g_M} \right\|_{L^\infty(I_{3/4, M+2})} \left(E_2 + \|g'\|_{L^\infty(I_{3/4, M+2})} \left\| 1 - \frac{g_M}{g} \right\|_{L^\infty(I_{3/4, M+2})} \right) \\ &\leq \left\| \frac{1}{g_M} \right\|_{L^\infty(I_{3/4, M+2})} (K(M^2 + 1) + 1) \max(E_1, E_2). \end{aligned}$$

We can use Theorem 4 with this to find that

$$\begin{aligned} E_5 &:= \left\| R_M(k) - \int_0^k \operatorname{Im} \left(\frac{g'(\frac{t}{s})}{g(\frac{t}{s})} \right) dt \right\|_{l^\infty(i_{3/4, M})} \\ &\leq M \left\| Q_M(k) - \int_{k-1}^k \operatorname{Im} \left(\frac{g'(\frac{t}{s})}{g(\frac{t}{s})} \right) dt \right\|_{l^\infty(i_{3/4, M} \setminus \{1 - \lfloor \frac{3}{4}(M-1) \rfloor\})} \\ &\leq M \left(\left\| \frac{g'_M(\frac{k}{Ms})}{g_M(\frac{k}{Ms})} - \frac{g'(\frac{k}{Ms})}{g(\frac{k}{Ms})} \right\|_{l^\infty(j_{3/4, M})} E_4 + C_2(2M/\delta)^{1/2} e^{-\frac{\pi\delta M}{8}} \left\| \frac{g'(\frac{k}{Ms})}{g(\frac{k}{Ms})} \right\|_{l^\infty(j_{3/4, M})} \right) \\ &\leq \left\| \frac{1}{g_M} \right\|_{L^\infty(I_{3/4, M+2})} (K(M^2 + 1) + 1) M^{3/2} E_2 + C_2 K M^{3/2} (2/\delta)^{-1/2} e^{-\frac{\pi\delta M}{8}}. \end{aligned}$$

We now put everything together. Using the elementary inequality $||u|^{1/2} e^{i\theta} - |v|^{1/2}| \leq |u - v|^{1/2} + |1 - e^{i\theta}| |v|^{1/2}$ along with (8) gives

$$\begin{aligned} E_6 &:= \left\| f_M \left(\frac{z}{s} \right) - \eta f \left(\frac{z}{s} \right) \right\|_{L^\infty(I_{1/4, M})} \\ &\leq \left\| f_M \left(\frac{k}{s} + ic \right) - \eta f \left(\frac{k}{s} + ic \right) \right\|_{l^\infty(i_{3/4, M})} E_3 + C_1 e^{-\frac{\pi}{8}(1-\frac{b}{s})(M-1)+2\pi c} \|f\|_{L^\infty(\mathbb{R})} \\ &\quad + \left\| \sum_{l \in i_{1, M} \setminus i_{3/4, M}} e^{-\frac{\pi}{2(M-1)}(z-l)^2 + \pi c} \right\|_{L^\infty(\mathbb{R})} \left\| f_M \left(\frac{k}{s} + ic \right) \right\|_{l^\infty(i_{1, M} \setminus i_{3/4, M})} \\ &\leq \left\| \left| g_M \left(\frac{k}{s} \right) \right|^{1/2} e^{\frac{i}{2}(R_M(k) + \arg g_M(0))} - \eta f \left(\frac{k}{s} + ic \right) \right\|_{l^\infty(i_{3/4, M})} E_3 \\ &\quad + 2C_1 e^{-\frac{\pi}{8}(1-\frac{b}{s})(M-1)+4\pi c} \|f\|_{L^\infty(\mathbb{R})} \end{aligned}$$

$$\begin{aligned}
&\leq \left(E_1^{1/2} + \left\| 1 - \exp \left(R_M(k) - \int_0^k \operatorname{Im} \left(\frac{g'(t)}{g(t)} \right) dt + \arg g_M(0) - \arg g(0) \right) \right\|_{l^\infty(i_{3/4,M})} \right. \\
&\quad \cdot \|f\|_{L^\infty(\mathbb{R})} E_3 \Big) + 2C_1 e^{-\frac{\pi}{8}(1-\frac{b}{s})(M-1)+4\pi c} \|f\|_{L^\infty(\mathbb{R})} \\
&\leq \left(E_1^{1/2} + \frac{1}{2}(E_5 + E_1) \|f\|_{L^\infty(\mathbb{R})} \right) E_3 + 2C_1 e^{-\frac{\pi}{8}(1-\frac{b}{s})(M-1)+4\pi c} \|f\|_{L^\infty(\mathbb{R})} \\
&= C_3(f) M^{7/2} \exp \left(-\min \left(\frac{\pi}{16} \left(1 - \frac{2b}{s} \right), \frac{\pi\delta}{8} \right) M + \max \left(\frac{\pi c^2}{2M}, 4\pi c \right) \right)
\end{aligned}$$

for some $C_3(f) < \infty$ and sufficiently large M , which establishes the result. \square

5 Numerical Experiments

We illustrate how Algorithm 5 works on two test cases. We consider the translated Bessel function (see [15]) $f(z) = J_1(z+20)$ sampled at $z = k$, $-M \leq k \leq M$, and a collection of $2M+1$ samples taken from an 8-bit, 44khz audio file. To show the effect of the parameter c , we take $c = 0.1$ in the first example and $c = 0.04$ in the second one. Values of G^* are tabulated using the built-in Gauss-Kronrod algorithm in MATLAB. We measure the worst-case error over $I_{1/2,M+1}$ as defined in Section 4, in order to avoid influence from inaccuracies close to the boundary of the full domain $I_{1,M+1}$.

M	Error over $I_{1/2,M+1}$
10	$3.7490 \cdot 10^{-2}$
20	$5.9513 \cdot 10^{-4}$
30	$4.0158 \cdot 10^{-5}$
40	$3.8732 \cdot 10^{-6}$
50	$3.8362 \cdot 10^{-7}$

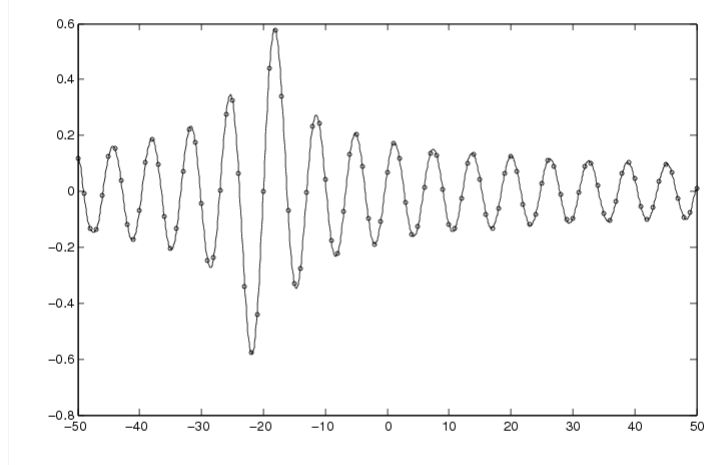


Figure 1: The graph of $f(z) = J_1(z+20)$ and the reconstruction errors with $c = 0.1$. It is clear that the error decays exponentially as M increases.

M	Error over $I_{1/2, M+1}$
10	$1.9752 \cdot 10^{-2}$
20	$5.1622 \cdot 10^{-3}$
30	$1.5710 \cdot 10^{-4}$
40	$1.1563 \cdot 10^{-4}$
50	$8.8637 \cdot 10^{-5}$

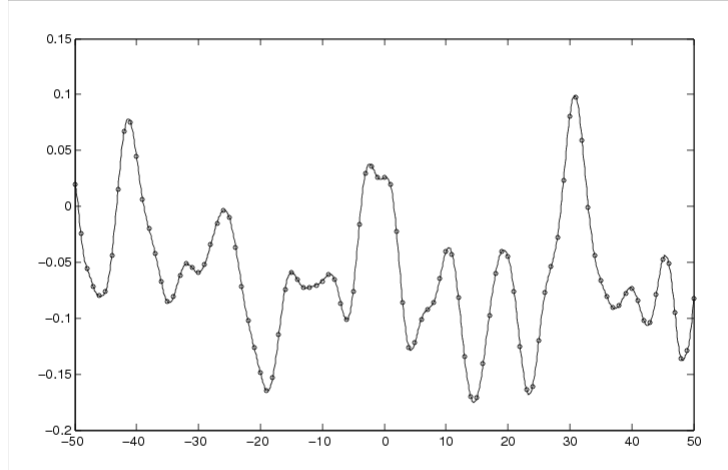


Figure 2: The graph of a section of audio data and the resulting reconstruction errors with $c = 0.04$. The convergence here is slower than it was in the preceding example. This is likely due to the presence of complex zeros with imaginary parts very close to c , as well as the proximity of the real zeros.

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